# Inverse Energy Cascade in Three-Dimensional Isotropic Turbulence

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We study the statistical properties of homogeneous and isotropic three-dimensional (3D) turbulent flows. By introducing a novel way to make numerical investigations of Navier-Stokes equations, we show that all 3D flows in nature possess a subset of nonlinear evolution leading to a reverse energy transfer: from small to large scales. Up to now, such an inverse cascade was only observed in flows under strong rotation and in quasi-two-dimensional geometries under strong confinement. We show here that energy flux is always reversed when mirror symmetry is broken, leading to a distribution of helicity in the system with a well-defined sign at all wave numbers. Our findings broaden the range of flows where the inverse energy cascade may be detected and rationalize the role played by helicity in the energy transfer process, showing that both 2D and 3D properties naturally coexist in all flows in nature. The unconventional numerical methodology here proposed, based on a Galerkin decimation of helical Fourier modes, paves the road for future studies on the influence of helicity on small-scale intermittency and the nature of the nonlinear interaction in magnetohydrodynamics.

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Inviscid invariants of the Navier-Stokes (NS) equations are crucial in determining the direction of the turbulent energy transfer [1]. In some cases, as for fully isotropic and homogeneous turbulence in 2D, the presence of two positively defined invariants (energy and enstrophy) does not allow a stationary transfer of both quantities, neither to small nor to large scales [2]. In the presence of two fluxes, they must necessarily flow in opposite directions [3–7] and this remains true even for turbulent systems in noninteger dimensions obtained by fractal Fourier decimation [8]. The fluid equations also possess two inviscid invariants in 3D: energy and helicity (i.e., the scalar product of velocity and vorticity). The inviscid conservation of helicity was discovered relatively recently [9,10]. At variance with energy, helicity is not positively defined. This allows for a simultaneous small-scale transfer of energy and helicity, as confirmed by the results of two-point closures [10–12] and direct numerical simulations [13,14]. Nevertheless, a reversal of the flux of energy has been observed in geophysical flows subject to the Earth's rotation [15,16] as well as in shallow fluid layers [17–22]. In both cases, this phenomenon is accompanied by strong anisotropic effects and by a substantial two-dimensionalization of the flow, induced either by the rotation or by the effects of confinement. Moreover, rotations inject fluctuations into the helical sector while a perfect two-dimensional flow has vanishing *pointwise* helicity, vorticity always being orthogonal to velocity. Here, we rationalize these findings, showing that inverse energy transfer is much broader than previously thought and is present in all flows in nature. In order to highlight this mechanism, we investigate in detail the transfer properties of NS equations in 3D homogeneous systems at changing the nature of the triadic nonlinear interactions. We show that an inverse energy cascade occurs also in 3D isotropic flow whenever parity invariance is broken and helicity acquires a well-defined sign at all wave numbers. The key new idea is to make a suitable surgery of the NS equations, such as to disentangle *triad by triad* the properties of the nonlinear energy transfer. In particular, we show that the energy flux is always reversed by keeping only triadic interactions between sign-defined helical modes, preserving homogeneity and isotropy and breaking reflection invariance. The role played by helicity in the energy transfer mechanism of 3D flows has attracted a broad scientific interest (see, e.g., [14] and references therein). Dynamical systems have been developed to study in detail energy and helicity transfer at high Reynolds numbers [23,24]. Further, speculations connecting the existence of intermittent bursts in the energy cascade induced by a "local" helicity blocking mechanism have been proposed [23]. Despite these important contributions, the understanding of the phenomenology of helicity remains "mysterious," as summarized in the conclusion of a recent state-of-the-art numerical study [14]. Here, we present theoretical and numerical evidence of a new phenomenon induced by helicity conservation: a statistically stationary *backward* energy transfer can be sustained even in 3D fully isotropic turbulence. The starting point of our analysis is the well-known helical decomposition [12] of the velocity field $\boldsymbol{v}(\boldsymbol{x})$, expanded in a Fourier series, $\boldsymbol{u}(\boldsymbol{k})$:

$$\boldsymbol{u}(\boldsymbol{k}) = u^{+}(\boldsymbol{k})\boldsymbol{h}^{+}(\boldsymbol{k}) + u^{-}(\boldsymbol{k})\boldsymbol{h}^{-}(\boldsymbol{k}), \tag{1}$$

 

where $\boldsymbol{h}^{\pm}$ are the eigenvectors of the curl operator $i\boldsymbol{k} \times \boldsymbol{h}^{\pm} = \pm k \boldsymbol{h}^{\pm}$. In particular, we choose $\boldsymbol{h}^{\pm} = \hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{k}} \pm i\hat{\boldsymbol{\nu}}$, where $\hat{\boldsymbol{\nu}}$ is an arbitrary versor orthogonal to $\boldsymbol{k}$ which satisfies the relation $\hat{\boldsymbol{\nu}}(\boldsymbol{k}) = -\hat{\boldsymbol{\nu}}(-\boldsymbol{k})$ (necessary to ensure the reality of the velocity field). Such a requirement is satisfied, e.g., by the choice $\hat{\boldsymbol{\nu}} = \boldsymbol{z} \times \boldsymbol{k}/\|\boldsymbol{z} \times \boldsymbol{k}\|$, with $\boldsymbol{z}$ an arbitrary vector. In terms of this *exact* decomposition of each Fourier mode energy, $E = \int d^3x |\boldsymbol{v}(\boldsymbol{x})|^2$, and helicity, $H = \int d^3x \boldsymbol{v} \cdot \boldsymbol{w}$, where $\boldsymbol{w}$ is the vorticity, are simultaneously diagonalized and written as

$$\begin{aligned} E &= \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H &= \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{aligned} \tag{2}$$

Similarly, the nonlinear term of the NS equations can be exactly decomposed in four independent classes of triadic interactions, determined by the helical content of the complex amplitudes, $\boldsymbol{u}^{s_k}(\boldsymbol{k})$, with $s_k = \pm$ (see [12] and right panel of Fig. 1). Among three generic interacting modes $\boldsymbol{u}^{s_k}(\boldsymbol{k})$, $\boldsymbol{u}^{s_p}(\boldsymbol{p})$, and $\boldsymbol{u}^{s_q}(\boldsymbol{q})$, one can identify eight different helical combinations ($s_k = \pm$, $s_p = \pm$, $s_q = \pm$). Among them, only four are independent because of the symmetry that allows us to change all signs of helicity simultaneously. Let us now consider the dynamics of an incompressible flow $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, which is determined by a decimated NS equation in which all interactions between modes have been switched off, except for those with a well-defined sign of helicity, e.g., positive ($s_k = +$, $s_p = +$, $s_q = +$) (class I in Fig. 1). We define the projector on positive or negative helicity states as

$$\mathcal{P}^{\pm} \equiv \frac{\boldsymbol{h}^{\pm} \otimes \overline{\boldsymbol{h}^{\pm}}}{\overline{\boldsymbol{h}^{\pm}} \cdot \boldsymbol{h}^{\pm}}, \tag{3}$$

where $\bar{\cdot}$ stands for the complex conjugate. Then, we project the velocity field into its positive helicity component

$$\boldsymbol{v}^+(\boldsymbol{x}) \equiv \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} \mathcal{P}^+ \boldsymbol{u}(\boldsymbol{k}) \tag{4}$$

and we consider the decimated NS equations

$$\partial_t \boldsymbol{v}^+ = (-\boldsymbol{v}^+ \cdot \boldsymbol{\nabla} \boldsymbol{v}^+ - \boldsymbol{\nabla} p)^+ + \nu \Delta \boldsymbol{v}^+ + \boldsymbol{f}^+, \tag{5}$$

where $\nu$ is the viscosity, $p$ is the pressure, and $\boldsymbol{f}$ is the external forcing stirring the fluid around a wave vector $k_f$. The nonlinear term and the forcing are projected on the positive helicity states, with the same procedure followed for the velocity field (4). The resulting system has two positive definite invariants—see Eq. (2)—the energy and the helicity, $H = \sum_k k|u^+(\boldsymbol{k})|^2$, and contains only interactions between positive helicity modes. Helicity becomes a coercitive quantity. Those interactions cannot sustain a simultaneous forward cascade of energy and helicity, for the same arguments which forbid the existence of a simultaneous forward cascade of energy and enstrophy in 2D turbulence [2,12]. Therefore, the dynamics of Eq. (5) should display a double cascade phenomenology, characterized by an inverse energy cascade with Kolmogorov spectrum $E(k) \sim k^{-5/3}$ for $k \ll k_f$ and a direct helicity cascade with a $k^{-7/3}$ spectrum for $k \gg k_f$. It is interesting to note that, at variance with usual 3D NS dynamics, such a flow should not display a dissipative anomaly for kinetic energy; i.e., energy dissipation should vanish in the limit

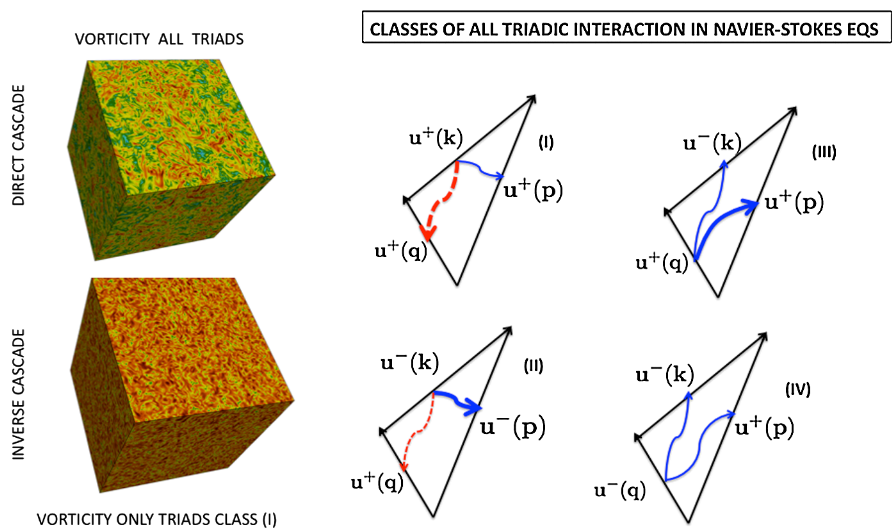


FIG. 1 (color online). Left: Comparison between vorticity fields for normal NS turbulence (top row) and for inverse cascade 3D turbulence (bottom row). Right: A pictorial scheme of all classes of triadic interactions in NS according to the helical Fourier decomposition and dynamical analysis proposed in [12]. Dashed (red) arrows denote backward energy transfer, while solid (blue) arrows stem for direct energy transfer. Thickness denotes different intensities of the energy flux. In particular, class I, the one investigated in this Letter, has backward events that dominate the dynamics, suggesting it as a candidate for the inverse cascade observed in different realistic 3D or quasi-3D flows configurations [13,15,19–21]; classes III and IV have only forward events, and class II has mixed events.

$\nu \to 0$. Indeed, the direct helicity cascade carries also a residual, nonconstant flux of kinetic energy toward small scales which decays as $k^{-1}$ and therefore vanishes in the high Reynolds number limit. As a consequence, one may speculate that the decimated NS equations possess a less singular spatiotemporal evolution and can be amenable to show the uniqueness and existence of solutions for all times. Numerical simulations have been performed with a fully dealiased, pseudospectral code at resolution $512^3$ on a triply periodic cubic domain of size $L = 2\pi$. The flow is sustained by a random Gaussian forcing, with $\langle f_i(\boldsymbol{k}, t) f_j(\boldsymbol{q}, t')\rangle = F(k)\delta(\boldsymbol{k} - \boldsymbol{q})\delta(t - t')Q_{i,j}(\boldsymbol{k})$, where $Q_{ij}(\boldsymbol{k})$ is a projector assuring incompressibility and $F(k)$ has support only in the high wave number range $|k| \in [k_{\min} = 25{:}k_{\max} = 32]$.

A visual inspection of the vorticity fields offered in Fig. 1 shows the differences between the forward cascade, which develops in standard 3D NS equations forced at large scales, and the novel 3D inverse cascade regime obtained from the decimated NS Eq. (5) forced at small scales. The latter does not possess any filamentary structure in the vorticity field, witnessing the fact that the vortex stretching mechanism, which is responsible for the forward cascade in standard 3D systems, is here absent.

In Fig. 2, we show a typical evolution of the energy spectrum obtained from Eq. (5) by initializing the flow with energy only at high wave numbers. The development of an inverse cascade with a Kolmogorov spectrum $E(k) \sim k^{-5/3}$ is unambiguous.

In the absence of a large-scale dissipative mechanism, the inverse cascade would accumulate the kinetic energy in the lowest available mode, originating a *condensed state* [20]. In order to avoid this phenomenon, we made a second series of numerical simulations, adding a hypoviscosity at large scales $\propto \Delta^{-1}\boldsymbol{v}$. In such a case, the total kinetic energy becomes stationary, as shown in Fig. 3, and is equally distributed among the three velocity components, showing that the flow is fully isotropic. This allows us to study scaling properties without having to cope with anisotropic subleading contributions [25]. In the inset of Fig. 3, we show the stationary energy flux in Fourier space, defined as $\Pi(k) \equiv (d/dt)\int_k^\infty E(p)dp$, where the time derivative is computed by taking into account only the nonlinear terms of Eq. (5). The negative plateau in the inertial range of wave numbers is a clear indication of the large-scale energy transfer.

The inverse cascade which arises from Eq. (5) is not intermittent. The probability distribution functions (PDFs) of the longitudinal velocity increments $\delta_r v = [\boldsymbol{v}(\boldsymbol{x} + \boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})] \cdot \hat{\boldsymbol{r}}$ at a distance $r$ within the inertial range are self-similar and almost Gaussian (see the inset of Fig. 4). The scaling of the second- and the fourth-order moments of velocity increments $S_2(r) = \langle(\delta_r v)^2\rangle$ and $S_4(r) = \langle(\delta_r v)^4\rangle$ follow the dimensional scaling $S_p(r) \sim r^{p/3}$ (see Fig. 4). This is a signature of all known inverse cascades (see, e.g., [26] for the case of a 2D NS equation) when fluctuations are transferred from faster to slower degrees of freedom [27]. Previous studies have shown the possibility to produce large-scale motion by nonparity invariant small-scale forcing only at small Reynolds numbers or in the quasi-linear regime [28]. Conversely, our results do not trivially originate from the projection of the forcing on the positive helicity states but are genuine effects of the nonlinear dynamics. To assess this issue, we performed a test simulation of the complete NS equation with the same projected forcing used in Eq. (5). After an initial transient, in which part of the energy accumulates at the forcing scale, a direct cascade sets in and all the energy injected is transferred toward small scales. This excludes the possibility that the forcing alone could be responsible for the inverse energy transfer observed in the decimated NS equation.

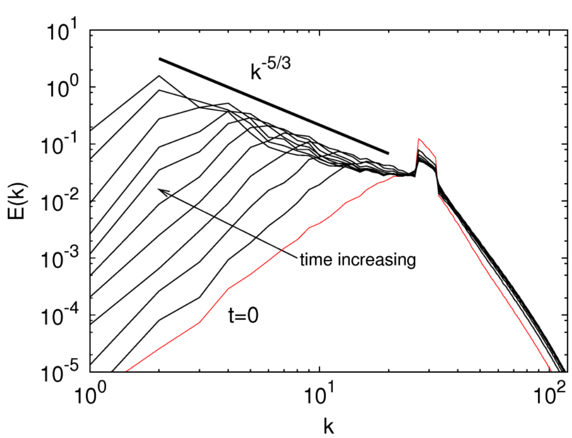


FIG. 2 (color online). Nonstationary spectrum in the inverse energy cascade regime. The straight dashed line represents the $k^{-5/3}$ slope.

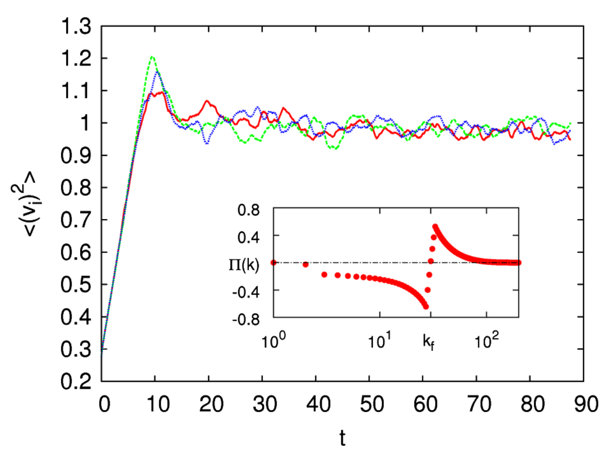


FIG. 3 (color online). Evolution of the three components of the turbulent kinetic energy as a function of time, $\langle(v_i)^2\rangle$, with $i = x$ [solid (red) line], $i = y$ [dashed (green) line], and $i = z$ [dotted (blue) line]. Inset: energy flux, $\Pi(k)$, in the Fourier space. Notice the clear negative plateau in the inertial range $k < k_f$.

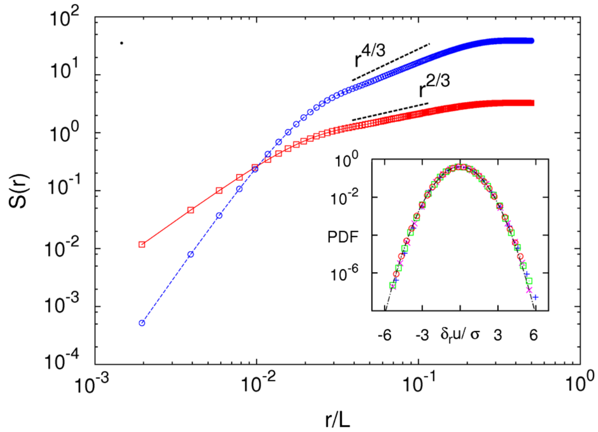


FIG. 4 (color online). Second-order (squares) and fourth-order (circles) structure functions in real space. Black solid and dashed lines represent the dimensional scaling $\propto r^{2/3}$ and $r^{4/3}$, respectively. Inset: PDFs of $\delta_r v$ at various scales $r = [1/4, 1/8/, 1/16, 1/32]L$, compared with the Gaussian distribution (dash-dotted line). Notice the perfect rescaling, supporting the absence of intermittency.

In conclusion, we have presented theoretical and numerical evidence that a *decimated* version of the NS equations, such that only modes with a given sign of helicity are retained, displays an inverse energy transfer mechanism. This phenomenon, which has been previously observed only in 2D turbulence or in strongly anisotropic 3D flows under bidimensionalization effects, is here observed for the first time in a fully isotropic 3D system and is intrinsically connected to the nonlinear dynamics of *all* flows in nature. Our findings show that *all 3D flows* in nature possess nonlinear interactions that may lead to a statistically stationary inverse energy cascade.

The scientific impact of our findings is manifold. First, it allows us to highlight those backward events in the energy transfer mechanism which are known to exist also in *untruncated* NS equations and that are one of the main theoretical and applied challenges; see, e.g., [29] for the case of subgrid modeling in large eddy simulations. Second, the link between backward energy events with the helical nature of triad interaction shows the key role of the coupled energy-helicity dynamics. Third, by clearly detecting which triadic interaction is responsible for forward and backward energy transfer, we pave the road for closure and analytical approaches aimed at understanding the whole energy transfer distribution.

This Letter also opens the way to further investigations. An obvious extension would be to integrate Eq. (5) with a large-scale forcing. In this case, a pure forward helicity cascade must develop as recently observed in flows of geophysical interest [13], provided that energy is removed at the forcing scale to avoid a pileup of fluctuations. More interesting, one could consider the case of a complementary decimation with respect to the one discussed here, i.e., eliminating only those triads that transfer energy backward (classes III and IV in Fig. 1). It is very tempting to speculate that such a system could display a direct energy cascade with reduced—or even vanishing—intermittency because one has removed all the *obstacles*, i.e., those events in which the forward energy transfer is stopped and/or reversed by the interaction with the helicity flux [23]. A surgery of interactions is potentially a tool to gauge the degree of small-scale intermittency as a function of the nature of the triads; it opens a new methodology for theoretical and numerical studies of statistical turbulence. Finally, similar decomposition may shed lights also in the evolution of conducting fluids, where three invariants, kinetic plus magnetic energy, cross helicity, and magnetic helicity, are known to produce a complex phenomenology [30].

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